STIFFNESS OF LIGHTWEIGHT CEMENTED SOILS: HOMOGENIZATION SCHEMES VS EXPERIMENTAL DATA

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Abstract

The derivation of stiffness modulus of Lightweight Cemented Soils (LWCS) through a micromechanical approach has been performed by implementing analytical formulae founded on the Mean-Field Eshelby-based Homogenization schemes. For the purpose of homogenization, the artificial porosity induced by the addition of foam has been evaluated by means of X-Ray micro-CT scans. The bulk modulus of the cemented matrix has been derived by mechanical tests performed on LWCS samples. The homogenized stiffness modulus has been computed for different curing times and the results are in agreement with experimental tests.

1. Composite materials and the concept of R.V.E

The homogenization theory aims at estimating the effective behavior of composite materials. The main interest of the approach lies on the possibility to use the obtained effective behavior to perform computations at the scale of the homogeneous structure by reasoning on the so-defined homogenized structure instead over the original heterogeneous one [1].

In many materials of interest, the microstructure can be considered as ***Statistically Homogeneous (S.H)***, i.e., the statistical descriptors of the geometrical arrangement do not depend on the position they are evaluated at. For such systems, it makes sense to define volume averaged properties, which are then independent of the size and position of the volume element considered, provided it is sufficiently large. A volume element that contains all the necessary information for the statistical description of a given microstructure is called ***Reference Volume Element*** **(R.V.E.)** [2].

The R.V.E. must comply with two conditions:

- to be elementary, which means that it is small enough compared to the size L of the structure;

- to be representative, that is to be large enough compared to the size d characterizing the heterogeneity of the microstructure [3].

In general, quasi-homogeneous sub-domains with known physical quantities (such as volume fraction, elastic or strength properties) are reasonably defined within the R.V.E., in order to represent the microstructure within each R.V.E., which is complicated to describe in detail. These sub-domains are referred to as *material phases*. The central objective of continuum micromechanics is to estimate the mechanical properties of the material defined on the R.V.E. from the aforementioned phase properties [1].

One strategy for modeling the mechanical behavior of microstructured materials consists in approximating the actual stress and strain fields by phase-wise constant fields, the statistics of the microgeometry being accounted for in terms of simple descriptors such as the phase volume fractions and the overall symmetry. This modeling philosophy leads to Mean Field estimates, Hashin-Shtrikman estimates, and Hashin-Shtrikman-type bounds, which in their basic forms provide analytical results for the linear behavior of inhomogeneous materials [4].

2. Mean-Field approaches

The more sophisticated Mean-Field Homogenizations (M.F.H) are based on Eshelby's solution. In the mean field methods, well adapted for heterogeneous materials with a random microstructure distribution, average fields are considered for each phase in the material [5]. M.F.H. is used to model composites with one matrix phase and one or multiple inclusion phases with uniform properties for each phase. These approaches require only partial information of the microstructure including the volume fraction, aspect ratio and the orientation of the inclusions [6].

In particular, mean-field approaches (M.F.As) approximate the micro-fields within each constituent by their volume phase averages $\underline{\underline{\boldsymbol{\varepsilon}}}_{\mu}^{(p)}$ and $\underline{\underline{\boldsymbol{\sigma}}}_{\mu}^{(p)}$, i.e., uniform strain and stress fields on each phase are used. The main geometrical characteristics of each phase, given by the volume fraction of each constituent, phase topology, aspect ratio of inclusions, etc., are considered by using statistical descriptors. In M.F.As the relations between the micro- and macro-fields are given by the following expressions (the dependence on the macroscopic coordinate $\underline{\boldsymbol{x}}$ is omitted for clarity) [7]:

$$\underline{\underline{\boldsymbol{\varepsilon}}}_{\mu}^{(p)} = \overline{\underline{\underline{\underline{\underline{\boldsymbol{A}}}}}}^{(p)} : \underline{\underline{\boldsymbol{\varepsilon}}}$$

$$\underline{\underline{\boldsymbol{\sigma}}}_{\mu}^{(p)} = \overline{\underline{\underline{\underline{\underline{\boldsymbol{B}}}}}}^{(p)} : \underline{\underline{\boldsymbol{\sigma}}}$$

Equation 1

and the homogenization relations can be written as:

$$\underline{\underline{\boldsymbol{\varepsilon}}}_{\mu}^{(p)} = \frac{1}{V_{\mu}^{(p)}} \int_{\Omega_{\mu}^{(p)}} \underline{\underline{\boldsymbol{\varepsilon}}}_{\mu} \left(\underline{\boldsymbol{y}}, t\right) dV \;\; with \; \underline{\underline{\boldsymbol{\varepsilon}}}_{\boldsymbol{\mu}} = \sum_{p} c^{(p)} \underline{\underline{\boldsymbol{\varepsilon}}}_{\mu}^{(p)}$$

$$\underline{\underline{\boldsymbol{\sigma}}}_{\mu}^{(p)} = \frac{1}{V_{\mu}^{(p)}} \int_{\Omega_{\mu}^{(p)}} \underline{\underline{\boldsymbol{\sigma}}}_{\mu} \left(\underline{\boldsymbol{y}}, t\right) dV \;\; with \; \underline{\underline{\boldsymbol{\sigma}}}_{\boldsymbol{\mu}} = \sum_{p} c^{(p)} \underline{\underline{\boldsymbol{\sigma}}}_{\mu}^{(p)}$$

Equation 2

Where p denotes a given phase of the material, $V^{(p)}$ is the volume occupied by this phase, $c^{(p)} = V^{(p)}/\sum_N V^{(k)}$ is the volume fraction of the phase (it is easy to see that $\sum_N c^{(p)} = 1$) and $\underline{\boldsymbol{y}}$ is the microscopic coordinate. It must be noticed that the phase concentration tensors $\overline{\boldsymbol{A}}^{(p)}$ and $\overline{\boldsymbol{B}}^{(p)}$ used in M.F.As are not functions of the spatial coordinates of the microstructure and they are considered to be constant over each phase.

In a first step, a local problem for a single inclusion is solved in order to obtain approximations for the local field behavior as was derived by Eshelby for elastic fields of an ellipsoidal inclusion. The second step consists on averaging the local fields to obtain the global effective properties [7].

For the generic phase p, $\overline{\boldsymbol{A}}^{(p)}$ is a fourth-order localization or concentration tensor of the p-phase (as it concentrates a macroscopic quantity prescribed at the boundary into a microscopic phase).

The most common one that applies best to the morphology encountered at all different levels of cement-based materials, is the Eshelbian-type ellipsoidal inclusion embedded in a reference medium for which an estimate $\overline{\boldsymbol{A}}^{(p)est}$ of the localization tensor is given in the form [8]:

$$\overline{\boldsymbol{A}}^{(p)est} = \left[\boldsymbol{I} + \boldsymbol{S}_{\boldsymbol{p}}^{\boldsymbol{Esh}}:(\boldsymbol{C}_{\boldsymbol{0}}^{-1}:\boldsymbol{C}_{\boldsymbol{p}} - \boldsymbol{I})\right]^{-1} : \langle\left[\boldsymbol{I} + \boldsymbol{S}_{\boldsymbol{p}}^{\boldsymbol{Esh}}:(\boldsymbol{C}_{\boldsymbol{0}}^{-1}:\boldsymbol{C}_{\boldsymbol{p}} - \boldsymbol{I})\right]^{-1}\rangle_V^{-1}$$

Equation 3

Where $\boldsymbol{C}_{\boldsymbol{0}}$ is the tensor of elastic moduli of the reference medium, $\boldsymbol{C}_{\boldsymbol{p}}$ is the fourth-order elasticity tensor of phase p=1,…,n, and $\boldsymbol{S}_{\boldsymbol{p}}^{\boldsymbol{Esh}}$ is the Eshelby tensor of phase p, which depends on $\boldsymbol{C}_{\boldsymbol{0}}$, the geometry, and the orientation of phase p. Given the random microstructure of cement-based materials, it is naturally to consider all phases as isotropic and the inclusions as spherical. The first assumption implies the isotropy of the local and the reference medium, that is [8]:

$$\boldsymbol{C}_{\boldsymbol{p}} = 3k_p\boldsymbol{J} + 2\mu_p\boldsymbol{K}$$

$$\boldsymbol{C}_{\boldsymbol{0}} = 3k_0\boldsymbol{J} + 2\mu_0\boldsymbol{K}$$

Equation 4

Where k_p, μ_p, k_0 and μ_0 are the bulk moduli and the shear moduli of phase r and of the reference medium, respectively; $J_{ijkl} = (1/3)\,\delta_{ij}\delta_{kl}$ is the volumetric part of the fourth-order unit tensor $\boldsymbol{I}$, and $\boldsymbol{K} = \boldsymbol{I} - \boldsymbol{J}$ is the deviatoric part.

For spheroidal inclusions (i.e., ellipsoids of rotation) in an isotropic elastic matrix, $\boldsymbol{S}_{\boldsymbol{p}}^{\boldsymbol{Esh}}$ can be evaluated analytically and depends only on the Poisson's ratio of the homogeneous material (or, in the case of inhomogeneous inclusions, on the Poisson's ratio of the matrix) and on the aspect ratio a of the inclusion. Therefore, the second assumption of spherical inclusions implies the following form of the Eshelby tensor [9]:

$$\boldsymbol{S}^{\boldsymbol{0}\ \boldsymbol{Esh}} = \alpha^0\boldsymbol{J} + \beta^0\boldsymbol{K} \quad \boldsymbol{I} = \boldsymbol{J} + \boldsymbol{K}$$

Equation 5

Where:

$$\alpha_0{}^{est} = \frac{3k^0}{3k^0 + 4\mu^0} \quad \beta_0{}^{est} = \frac{6(k^0 + 2\mu^0)}{5(3k^0 + 4\mu^0)}$$

Equation 6

With μ^0 and k^0 the shear moduli and the bulk moduli of the reference medium, respectively. Notice that tensor itself does not depend on the radius of the sphere [9].

To homogenize the local material properties, constitutive relations are required for the different phases, together with the volume-averaging relation linking the microscopic and the macroscopic stress [8]:

$$\underline{\underline{\boldsymbol{\sigma}}} = \langle \underline{\underline{\boldsymbol{\sigma}}}_{\boldsymbol{\mu}} \left(\underline{\boldsymbol{y}}\right) \rangle_{\boldsymbol{V}}$$

Equation 7

Use in Equation 7 of a linear elastic constitutive law for each microscopic phase, i.e., of the [8]:

$$\underline{\underline{\boldsymbol{\sigma}}}_{\mu}^{(p)} = \underline{\underline{\underline{\boldsymbol{C}}}}_{\boldsymbol{p}} : \underline{\underline{\boldsymbol{\varepsilon}}}_{\mu}^{(p)}$$

Equation 8

together with the strain localization condition (Equation 1), delivers the following linear homogenization formula for the macroscopic (or homogenized) elasticity tensor $\underline{\underline{\underline{\boldsymbol{C}}}}_{\boldsymbol{hom}}$ [8]:

$$\underline{\underline{\boldsymbol{\sigma}}} = \underline{\underline{\underline{\boldsymbol{C}}}}_{\boldsymbol{hom}} : \underline{\underline{\boldsymbol{\varepsilon}}}$$

Equation 9

$$\underline{\underline{\underline{\boldsymbol{C}}}}_{\boldsymbol{hom}} = \langle \underline{\underline{\underline{\boldsymbol{C}}}}_{\boldsymbol{p}} : \underline{\underline{\underline{\boldsymbol{A}}}}_{\boldsymbol{p}} \rangle_{\boldsymbol{V}} = \sum_{\boldsymbol{r}} c_p \underline{\underline{\underline{\boldsymbol{C}}}}_{\boldsymbol{p}} : \underline{\underline{\underline{\boldsymbol{A}}}}_{\boldsymbol{p}}$$

Equation 10

While Equation 10 is an exact theoretical definition of $\underline{\underline{\underline{\boldsymbol{C}}}}_{\boldsymbol{hom}}$, the practical determination of $\underline{\underline{\underline{\boldsymbol{C}}}}_{\boldsymbol{hom}}$ is generally based on estimates of the localization tensor for each phase $\underline{\underline{\underline{\boldsymbol{A}}}}_{\boldsymbol{p}}{}^{\boldsymbol{est}}$. It is readily understood that the quality of the homogenization result is intimately related to the quality of the localization condition. In a refined analysis, considering the Eshelbian-type strain localization (Equation 3), the following estimate of the macroscopic (or homogenized) elasticity tensor $\underline{\underline{\underline{\boldsymbol{C}}}}_{\boldsymbol{hom}}{}^{\boldsymbol{est}}$ is obtained [8]:

$$\underline{\underline{\underline{\boldsymbol{C}}}}_{\boldsymbol{hom}}{}^{\boldsymbol{est}} = \langle \underline{\underline{\underline{\boldsymbol{C}}}}_{\boldsymbol{p}} : [\underline{\underline{\underline{\boldsymbol{I}}}} + \underline{\underline{\underline{\boldsymbol{S}}}}_{\boldsymbol{p}}^{\boldsymbol{Esh}} : (\underline{\underline{\underline{\boldsymbol{C}}}}_{\boldsymbol{0}}{}^{-\boldsymbol{1}} : \underline{\underline{\underline{\boldsymbol{C}}}}_{\boldsymbol{p}} - \underline{\underline{\underline{\boldsymbol{I}}}})]^{-\boldsymbol{1}} \rangle_{\boldsymbol{V}} : \langle [\underline{\underline{\underline{\boldsymbol{I}}}} + \underline{\underline{\underline{\boldsymbol{S}}}}_{\boldsymbol{p}}^{\boldsymbol{Esh}} : (\underline{\underline{\underline{\boldsymbol{C}}}}_{\boldsymbol{0}}{}^{-\boldsymbol{1}} : \underline{\underline{\underline{\boldsymbol{C}}}}_{\boldsymbol{p}} - \underline{\underline{\underline{\boldsymbol{I}}}})]^{-\boldsymbol{1}} \rangle_{\boldsymbol{V}}{}^{-\boldsymbol{1}}$$

Equation 11

Substituting Equation 4-Equation 6 into Equation 11 yields explicit expressions for the homogenized bulk modulus and shear modulus [8]:

$$\underline{\underline{\underline{\boldsymbol{C}}}}_{\boldsymbol{hom}}^{\boldsymbol{est}} = 3k_{hom}^{est} \underline{\underline{\underline{\boldsymbol{J}}}} + 2\mu_{hom}^{est} \underline{\underline{\underline{\boldsymbol{K}}}}$$

Equation 12

$$k_{hom}^{est} = \sum_r c_p k_p \left(1 + \alpha_0^{est}\left(\frac{k_p}{k_0} - 1\right)\right)^{-1} \times \left[\sum_r c_p \left(1 + \alpha_0^{est}\left(\frac{k_p}{k_0} - 1\right)\right)^{-1}\right]^{-1}$$

Equation 13

$$\mu_{hom}^{est} = \sum_r c_p \mu_p \left(1 + \beta_0^{est}\left(\frac{\mu_p}{\mu_0} - 1\right)\right)^{-1} \times \left[\sum_r c_p \left(1 + \beta_0^{est}\left(\frac{\mu_p}{\mu_0} - 1\right)\right)^{-1}\right]^{-1}$$

Equation 14

To close the upscaling procedure, we need to choose the appropriate reference medium, in which the inclusions are embedded. The Mori–Tanaka scheme (M.T.), in which the matrix phase is chosen as reference medium, i.e., $\underline{\underline{\boldsymbol{C}}}_{\boldsymbol{0}} = \underline{\underline{\boldsymbol{C}}}_{\boldsymbol{m}}$, is appropriate for materials that exhibit a strong matrix-inclusion morphology. This scheme is chosen for the two-phase spherical inclusion composites, for which Equation 13 and Equation 14 reduce to [8]:

$$\frac{k_{hom}^{est}}{k_m} = 1 + c_I \frac{{}^{k_I}/_{k_m} - 1}{1 + \alpha_m^{est}(1 - c_I)\left({}^{k_I}/_{k_m} - 1\right)}$$

Equation 15

$$\frac{\mu_{hom}^{est}}{\mu_m} = 1 + c_I \frac{\mu_I - \mu_m}{1 + \beta_m^{est}(1 - c_I)\left({}^{\mu_I}/_{\mu_m} - 1\right)}$$

Equation 16

$$\alpha_0^{est} \equiv \alpha_m^{est} = \frac{3k_m}{3k_m + 4\mu_m} \; and \; \beta_0^{est} \equiv \beta_m^{est} = \frac{6(k_m + 2\mu_m)}{5(3k_m + 4\mu_m)}$$

Equation 17

The homogenized Young’s modulus and Poisson’s ratio are evaluated from [8]:

$$E_{hom}^{est} = \frac{9k_{hom}^{est}\mu_{hom}^{est}}{3k_{hom}^{est} + \mu_{hom}^{est}} \; and \; \nu_{hom}^{est} = \frac{3k_{hom}^{est} - 2\mu_{hom}^{est}}{6k_{hom}^{est} + 2\mu_{hom}^{est}}$$

Equation 18

Let’s consider a R.V.E. made of an isotropic linear elastic matrix having a Poisson’s ratio ν_m of 0.2 and containing stress free spherical cavities having a volume fraction or a macroporosity p. Assume that the overall effective behavior of this porous material is isotropic. By setting $k_i = \mu_i = 0$ and using the Mori-Tanaka scheme, the effective normalized bulk and shear modulus are therefore given by the same following hyperbolic form [10]:

$$\frac{k_{M-T}^{h}(p)}{k_m} = \frac{\mu_{M-T}^{h}(p)}{\mu_m} = \frac{1 - p}{1 + p}$$

Equation 19

3. Results and conclusions

Three different homogenization schemes have been used to compute the homogenized stiffness modulus of LWCS samples, i.e., Mori-Tanaka (MT), differential method (DF), dilute method (DI). For each scheme, it has been shown that, under the hypothesis $v_m = 0.2$, the normalized bulk, shear and Young's moduli are given by the same form depending only on the porosity p. Then, the different models predictions have been compared with experimental Young's modulus, obtained by shear tests and uniaxial and isotropic compression tests. This comparison shows that the Mori-Tanaka method is the best homogenization scheme for LWCS. Therefore, the whole expressions (i.e., without the hypothesis $v_m = 0.2$) of Mori-Tanaka scheme have been employed to compute the homogenized stiffness modulus for different curing time of the LWCS samples.

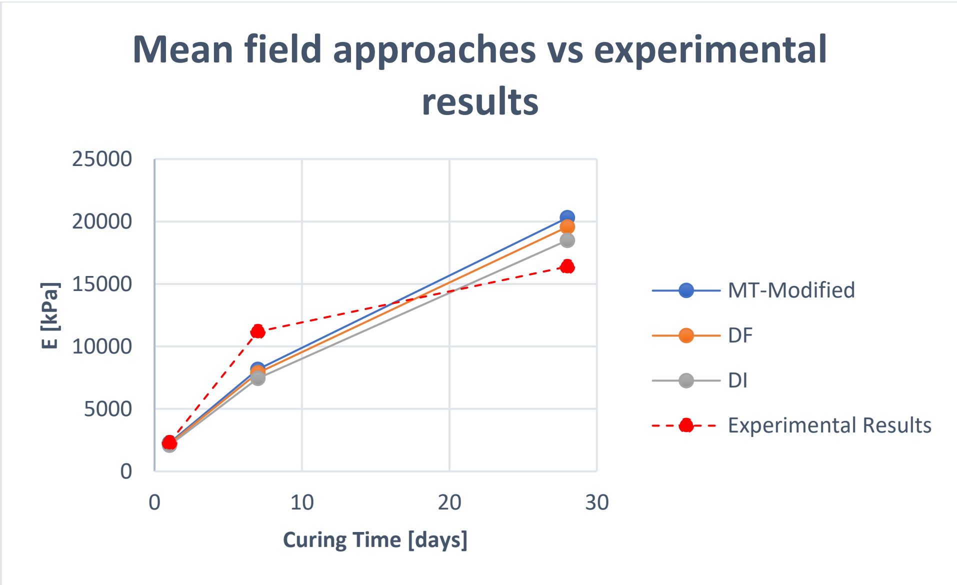

Figure 1: Mean Field approaches vs experimental results.

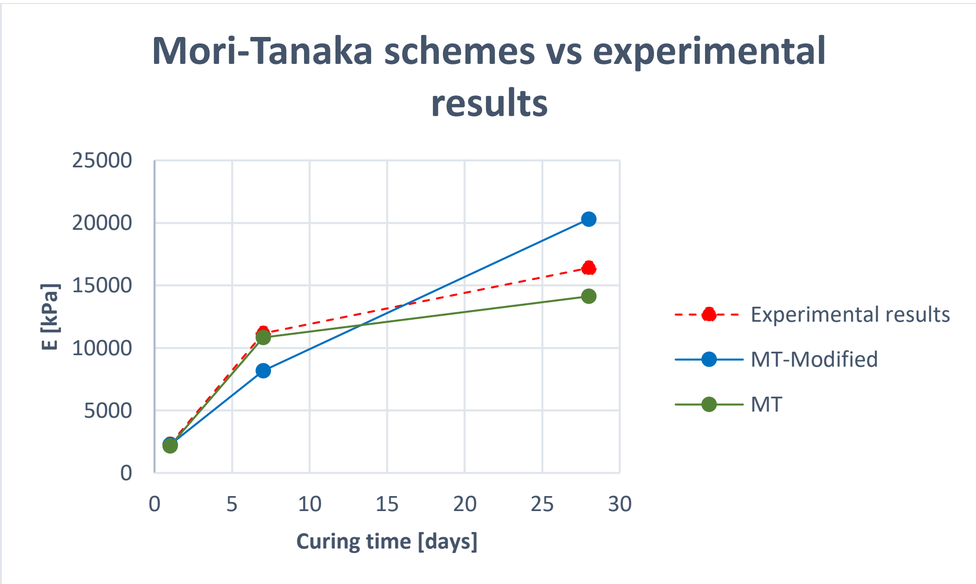

Figure 2: MT-schemes vs experimental results.

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